It will be convenient to use all of the notation of symbolic logic, so we can see more easily our application of DeMorgan's Law.

- $\vee$ OR
- $\wedge$ AND
- $\neg$ NOT
- $\Rightarrow$ IMPLIES
- $\Leftrightarrow$ IF AND ONLY IF

In this notation, DeMorgan's Laws may be written as follows.

- $\neg(p \wedge q) \Leftrightarrow [\neg p \vee \neg q]$
- $\neg(p \vee q) \Leftrightarrow [\neg p \wedge \neg q]$

Problem 1. Let A and B be sets. Show that

$$(A \smallsetminus B) \smallsetminus C = A \smallsetminus (B \cup C).$$

Solution 1. To show that two sets are equal, we show that each is contained in the other.

($\subset$) Let $x \in (A \smallsetminus B) \smallsetminus C$. Then $x \in A \smallsetminus B$ and $x \notin C$. Thus $x \in A$, $x \notin B$, and $x \notin C$. Now, by DeMorgan's Law,

$$x \notin B \wedge x \notin C \;\Leftrightarrow\; \neg(x \in B) \wedge \neg(x \in C) \;\Leftrightarrow\; \neg(x \in B \vee x \in C) \;\Leftrightarrow\; \neg(x \in B \cup C) \;\Leftrightarrow\; x \notin B \cup C,$$

so $x \in A$ and $x \notin B \cup C$. Thus $x \in A \smallsetminus (B \cup C)$.

($\supset$) Let $x \in A \smallsetminus (B \cup C)$. Then $x \in A$, and $x \notin B \cup C$. By DeMorgan's Law (as above), $x \notin B \cup C$ if and only if $x \notin B$ and $x \notin C$. So, $x \in A$ and $x \notin B$ and $x \notin C$. Then $x \in (A \smallsetminus B)$ and $x \notin C$, whence $x \in (A \smallsetminus B) \smallsetminus C$. □

Solution 2. We show that $x \in (A \smallsetminus B) \smallsetminus C$ if and only if $x \in A \smallsetminus (B \cup C)$:

$$\begin{aligned}
x \in (A \smallsetminus B) \smallsetminus C &\Leftrightarrow x \in (A \smallsetminus B) \wedge x \notin C \\
&\Leftrightarrow (x \in A \wedge x \notin B) \wedge x \notin C \\
&\Leftrightarrow (x \in A \wedge \neg(x \in B)) \wedge \neg(x \in C) \\
&\Leftrightarrow x \in A \wedge (\neg(x \in B) \wedge \neg(x \in C)) \\
&\Leftrightarrow x \in A \wedge \neg(x \in B \vee x \in C) \quad \text{(by DeMorgan's Law)} \\
&\Leftrightarrow x \in A \wedge \neg(x \in B \cup C) \\
&\Leftrightarrow x \in A \wedge x \notin B \cup C \\
&\Leftrightarrow x \in A \smallsetminus (B \cup C).
\end{aligned}$$

□